Term Premium Determinants of Three-Month Forward Interest Rate

Carlos Del Castillo* and Jean-François Fillion**

October 2002

Department of Finance Working Paper
2002-08

Department of Finance

Economic and Fiscal Policy Branch

The authors would like to thank for their comments, Patrick Georges and Pascal François (HEC, Montreal) and seminar participants at the Department of Finance, the 2001 meeting of the Société canadienne de sciences économiques and the 2001 meeting of the Northern Finance Association.

*delcastillo.carlos@fin.gc.ca Phone (613) 996-0319
**Fillion.Jean-Francois@fin.gc.ca Phone: (613) 992-6707

104-03313/

Abstract

Forward interest rates provide public authorities with insight into financial market expectations of future interest rates. Quite often the Expectation Hypothesis of the Term Structure (EHTS) is used to explain changes in forward interest rates. The EHTS assumes that forward rates, when adjusted for a term premium, are unbiased predictors of anticipated future interest rates. That said, it is well known that the EHTS is often rejected by the data, perhaps because the term premium varies over time.

In this paper, we develop a consumption-based asset pricing model aimed at explaining the term premium included in the three-month forward rate. In the model, the premium depends on two parameters — the degree of consumer's risk aversion and the persistence of consumption growth — and on one variable — the conditional variance of consumption growth. We then test the role of the conditional variance of consumption growth using an econometric equation for the term premium. The EHTS is not rejected in a broad equation that includes the conditional variance of consumption growth and a measure of the federal government debt-to-GDP ratio as a proxy for the time-varying risk premium often associated with changes in government debt. Using this equation, we find that the term premium for the three-month forward interest rate has dropped by about 50 basis points since its recent peak in 1995 following the decline in the government debt-to-GDP ratio.

Résumé

Les taux d'intérêt à terme contiennent de l'information utile pour les autorités publiques en ce qui a trait aux taux d'intérêt futurs anticipés par les marchés financiers. Pour expliquer l'évolution des taux d'intérêt à terme, il n'est pas rare d'utiliser l'hypothèse des attentes de la structure par terme des taux d'intérêt (HASTI). Cette hypothèse suppose que les taux à terme, lorsque ajustés pour la prime à terme, sont des mesures non-biaisées des taux d'intérêt futurs anticipés. Or, il est bien connu que l'HASTI est le plus souvent rejetée par les données, peut-être parce que la prime à terme est variable dans le temps.

Dans ce travail, nous développons un modèle théorique d'évaluation d'actif basé sur la consommation visant à expliquer la prime à terme comprise dans le taux à terme à trois mois. Dans le modèle, la prime dépend de deux paramètres – le degré d'aversion au risque du consommateur et le degré de persistance du taux de croissance de la consommation – ainsi que d'une variable, soit la variance conditionnelle du taux de croissance de la consommation. Nous vérifions ensuite le rôle de la variance conditionnelle de la consommation dans un équation économétrique de la prime à terme. L'HASTI n'est pas rejetée dans une équation élargie comprenant la variance conditionnelle du taux de croissance de la consommation et le ratio de la dette du gouvernement fédéral au PIB, lequel représente ici une mesure d'approximation pour les variations de la prime de risque souvent associées aux changements de l'endettement public. Avec cette équation, nous trouvons que la prime à terme du taux d'intérêt à terme à trois mois a baissé d'environ 50 points de base depuis son récent sommet de 1995 en raison de la diminution de l'endettement public.

Introduction

A good understanding of forward interest rates provides public authorities with insight into financial market expectations of future interest rates. This is helpful from the point of view of economic policy. Quite often the Expectation Hypothesis of the Term Structure (EHTS) is used to explain changes in forward interest rates. The EHTS assumes that forward rates, when adjusted for a term premium are unbiased predictors of anticipated future interest rates. That said, it is well known that the EHTS is often rejected by the data, perhaps because the term premium varies over time. However, because the term premium cannot be observed, we need a model to estimate it.

In this paper, we develop a theoretical model aimed at explaining the term premium included in the three-month forward interest rate. We use a consumption-based asset pricing model, assuming a representative investor who has a choice of consuming now or during future periods. Here, the premium depends on two parameters — the degree of consumer's risk aversion and the persistence of consumption growth — and on one variable, the conditional variance of (or the uncertainty associated with) consumption growth. With a reasonable value of the risk aversion coefficient, the model reproduces the average term premium observed in Canada since the 1960s.

However, in order to know if the theoretical model accurately reproduces the term premium variability, we must test it empirically. We develop an econometric equation for the term premium and assess the effect of the conditional variance of consumption growth. Although this variable is statistically significant, its presence in the model does not prevent rejection of the EHTS (i.e. the forward interest rate is an unbiased predictor of future interest rate). However, when we include in the equation other variables likely to explain changes in the term premium, such as a measure of the federal government debt-to-GDP ratio, the EHTS is not rejected.

In Section 1, which follows, we describe and test the EHTS. In Section 2, we present the theoretical model. In Section 3, we test the model empirically, using the econometric equation. Finally, in Section 4, we comment briefly on the results and discuss possible future extensions.

1. The Expectation Hypothesis of the Term Structure

The Expectation Hypothesis of the Term Structure (EHTS) is often rejected by the data, perhaps because the term premium varies over time, and changes to it are difficult to measure. In this section, we develop and test an econometric equation for the EHTS using Canadian three-month interest rates. This equation will also be used in Section 3 to test the validity of the theoretical model of the term premium.

The EHTS can be expressed on the basis of the forward interest rate as follows (Backus, Foresi, Mozumbar and Wu, 2001):

$$rf_{n,n+m,t} = E_t r_{m,t+n} + \varphi_m \qquad (1.1)$$

The variable $rf_{n,n+m,t}$, called the m-period forward interest rate, is the interest rate on a m-period forward contract starting at time n, maturing at time $n+m$, and contracted today at time t. Equation 1.1 indicates that according to the EHTS, the m-period forward rate is equal to the future m-period spot interest rate, expected at time t, n periods ahead ($E_t r_{m,t+n}$), plus a term premium φ_m. In the pure form of the EHTS, the term premium is equal to zero. In a weaker form of the EHTS (the liquidity preference theory), the term premium is positive and constant over time. In this paper, we make no prior assumptions about the structure of the term premium, allowing it to vary or not over time.

We focus on three-month interest rates (or one quarter, so that m=1) and a forecast horizon of three months (or one quarter, so that n=1). To simplify the notation, the three-month forward interest rate contract signed at time t and beginning in three months (n=1) is noted rf_{1t}, the expected three-month underlying spot interest rate is noted $E_t r_{1t+1}$, and the EHTS equation is written:

$$rf_{1t} = E_t r_{1t+1} + \varphi_1 \tag{1.2}$$

The EHTS equation can be modified, first by isolating all the terms on the right-hand side of the equation, by multiplying by -1, and then by adding $[r_{1t+1} - r_{1t}]$ to each side of the equation, with the following result:

$$[r_{1t+1} - r_{1t}] = [rf_{1t} - r_{1t}] - \varphi_1 + \mu_{t+1} \tag{1.3}$$

where $\mu_{t+1} = [r_{1t+1} - E_t r_{1t+1}]$.

This equation indicates that the change in the short-term interest rate between periods $t+1$ and t is equal to the term spread (the difference between the forward interest rate and the spot interest rate) adjusted for a term premium, plus a forecast error μ_{t+1} with a conditional mean of zero.

We can use the previous equation to test the validity of the EHTS empirically as follows:

$$[r_{1t+1} - r_{1t}] = \delta_0 + \delta_1 [rf_{1t} - r_{1t}] + \mu_{t+1} \tag{1.4}$$

If the EHTS holds (or is not rejected), $\delta_1 = 1$, and the term premium, here assumed to be constant, is equal to $-\delta_0$.

We can rewrite equation 1.4 to obtain an equation for the excess returns by subtracting $[rf_{1t} - r_{1t}]$ from each side of the equation and multiplying each side by -1, with the following result:

$$[rf_{1t} - r_{1t+1}] = -\delta_0 + (1 - \delta_1)[rf_{1t} - r_{1t}] - \mu_{t+1}$$

or

$$[rf_{1t} - r_{1t+1}] = \gamma_0 + \gamma_1[rf_{1t} - r_{1t}] - \mu_{t+1} \quad (1.5)$$

The term on the left-hand side of equation 1.5 represents the excess return, defined as the difference between the interest rate on the forward contract and the corresponding future spot interest rate. If the EHTS holds, $\gamma_1 = 0$ (or $\delta_1 = 1$), and the excess returns are independent of the term spread. In other words, if the term spread can be used to predict excess returns, the EHTS is rejected. To equation 1.5, we can also add other variables aimed at explaining the term premium included in excess returns. This is explored in Section 3.

Table 1 presents the results of EHTS tests using equation 1.5 for the sample period beginning in 1960 and ending in the third quarter of 2000 and for various sub-periods. We use quarterly data to avoid problems due to the moving average process in the estimation errors that occurs with this type of equation when data frequency (such as 12 times per year) is higher than the term of the investments (for example, three months, in our study).[1] The spot interest rate r_{1t} is the three-month Treasury bill rate (B14007); the forward interest rate rf_{1t} is the forward rate implicit in the yield curve extracted using the six-month (B14008) and the three-month Treasury bill rates.[2] It should also be noted that all estimated equations include a Generalized Autoregressive Conditional Heteroskedasticity (GARCH) process to take into account the nature of the errors frequently observed in financial data.[3]

The excess returns data and term spread are presented in Chart 1 (Appendix 3). The variability of the excess returns appears to have changed during the sample period, possibly because of changes in the term premium. Excess returns have been smaller and more stable since 1997; probably because this period has been marked by a healthier

[1] Econometric models with a complicated moving average process in the error term are often difficult to estimate with accuracy.
[2] We show in Appendix 1 how we can extract the three-month forward rate implicit in the yield curve under the assumption of absence of arbitrage.
[3] For better readability, the results of the GARCH process are not presented in detail.

macro-economic environment, with low inflation and a declining public debt relative to GDP.

The results presented in Table 1 indicate that the term spread $rf_{1t} - r_{1t}$ significantly helps predict excess returns throughout the sample period beginning in 1960, and thus the EHTS can be rejected over the whole sample. These results parallel those obtained by Hejazi, Lai and Yang (2000) with one-month interest rate data. If the sample period is divided into two sub-periods, a period of low variability in excess returns running from 1960 to 1979, and a period of greater variability starting in 1980, the EHTS is not rejected during the first sub-period, but it is rejected during the second. These results parallel those obtained by Paquette and Stréliski (1998) for the 1988-97 period. Table 1 also indicates that the EHTS is rejected during the more recent period of low inflation beginning in 1993, but not during the very recent period beginning in 1997.

Table 1
EHTS tests for various sub-periods
Estimated equation:
$$[rf_{1t} - r_{1t+1}] = \gamma_0 + \gamma_1 [rf_{1t} - r_{1t}] + \mu_{t+n}$$

	Estimated coefficients and z-statistics for various sub-periods				
	1960Q1-2000Q3	1960Q1-1979Q4	1980Q1-2000Q3	1993Q1-2000Q3	1997Q1-2000Q3
γ_0	0.0017	0.0021	0.0011	0.0011	0.0025
	(1.76)	(1.46)	(2.59)	(0.90)	(1.76)
$rf_{1t} - r_{1t}$	0.38	-0.35	0.67	0.51	0.19
	(2.20)	(-0.90)	(4.74)	(2.63)	(0.68)
Number of obs.	163	80	83	31	15

Overall, the EHTS (assuming a constant term premium) is rejected by the data during certain periods, but not in others. From a theoretical perspective, this ambiguity is not very satisfying. As has been noted, the rejection of the EHTS may be due to the assumption of a constant term premium. Since the unobservable term premium is likely

to vary over time, we need a model to estimate it. In the next section we present our theoretical model and then test it empirically in Section 3.

2. The theoretical model

Our theoretical model is an intertemporal consumption-based asset pricing model. This type of model has been used extensively by economists for modelling the equilibrium rates of return of various assets including the term premium of the term structure of interest rates. For example, Campbell (1986), Backus, Gregory and Zin (1989) and, more recently, Campbell and Viceira (2001) used this type of model to determine the theoretical value of the term premium based on different state variables in the economy, like consumption and interest rate uncertainty. However, all these papers assumed that the term premium is constant over time. Following the evidence of the rejection of the EHTS and the existence of a time-varying term premium, it appears to us that extending the theoretical model to allow for a time-varying premium is an important area of research.

Our theoretical model allows for a time-varying term premium and to that extent, it shares some similarities with the model developed by Lee (1995). In Lee's model, the variability of the term premium depends on the conditional variances and covariances of real output and money growth. However, in a recent empirical analysis of Lee's model with Canadian data, Hejazi, Lai and Yang (2000) did not find support for the role of these variables at explaining the term premium at the short-end of the term structure. In our theoretical model, the term premium will vary depending on the conditional variance of consumption growth. This variable will also be found to be significant at explaining the variability of the term premium using Canadian data.

2.1 Investor's optimization problem

The model relates the rates of return on securities maturing in one and two periods to consumption and saving decisions of a representative investor. First, we present the investor's optimization problem. Then, we derive the theoretical rates of return on those

securities. These rates of return are then used to establish the term premium of a one-period forward contract maturing in two periods.

The optimization problem of the representative investor is presented as follows:

$$U_t = \max_{\{C_{t+j},\omega_{t+j}\}_{j=0}^{\infty}} \left(\frac{1}{1-\alpha}\right) E_t \sum_{j=0}^{\infty} B^j C_{t+j}^{1-\alpha} \tag{2.1}$$

$$A_{t+1} = RP_t(A_t - C_t) \tag{2.2}$$

where E_t is the expectation operator conditional on information available at time t. U_t represents the utility at time t of current and expected consumption for the representative investor; C_{t+j} represents the investor's real consumption at time $t + j$; A_{t+1} represents the investor's real wealth at time $t + 1$; $RP_t = \omega_t R_{1t} + (1 - \omega_t)R_{2t}$ represents the total yield of the securities portfolio from time t to $t + 1$. The portfolio includes securities maturing in one period with a yield of R_{1t} or $(1+r_{1t})$ as well as securities maturing in two periods with a yield of R_{2t} or $(1+r_{2t})$, where r_{1t} and r_{2t} represent the rates of return of these securities and ω and $(1-\omega)$ represent the weight in the portfolio of these securities. The parameter α captures risk aversion or the reluctance to exchange a "sure" consumption for a "fair consumption gamble". The discount factor B measures the investor's patience (or time preference), that is, the investor's willingness to forego consumption at time t for consumption at time $t + j$.

Equation 2.1 describes the investor's preferences, which are iso-elastic. With this type of preference, risk aversion and the term premium do not vary over time when the economy is expanding. Equation 2.2 represents the budget constraint; it means simply that the individual's total wealth held at time $t+1$ (A_{t+1}) is equal to the wealth held during the previous period, adjusted by the amount of consumption $(A_t - C_t)$, which is invested in securities maturing in one and two periods with a total yield of RP_t. It should be noted that the model assumes that the investor consumes only the product of his initial holding

of financial wealth, and thus assumes no labour income and no uncertainty about labour income.[4]

The investor seeks to maximize the utility of consumption for all periods, knowing his risk aversion α and his time preference B. In order to determine the optimum consumption path, we use the first-order conditions of the intertemporal consumption and portfolio choice problem and derive the Euler equations, as follows:

$$1 = E_t\left[B^j \left(\frac{C_{t+j}}{C_t} \right)^{-\alpha} (R_{a,j})^j \right] \tag{2.3}$$

The term $B^j \left(\frac{C_{t+j}}{C_t} \right)^{-\alpha}$ represents the marginal rate of intertemporal substitution between periods t and $t+j$. This rate is also known as the stochastic discount factor, and is used to measure the value of the portfolio and individual securities, including those maturing in one and two periods. This formulation ensures that two securities with the same systematic risk produce the same expected return, reflecting the absence of arbitrage. An implication of equation 2.3 is also that the higher the investor's risk aversion, the less intertemporal substitution that investor will make.

2.2 Determining one- and two-period rates of return

Securities maturing in one period have a rate of return of r_{1t} and their yield is defined by R_{1t} or $(1+ r_{1t})$. At maturity, the security is worth $P_{1t+1} = 1$, assuming that it is purchased at discount, such as a Treasury bill, and produces no dividends ($D_{1t+1} = 0$). The purchase price P_{1t} determines the yield of the security between time t and $t+1$.

$$R_{1t} = 1 + r_{1t} = \frac{P_{1t+1} + D_{1t+1}}{P_{1t}} = \frac{1}{P_{1t}} \tag{2.4}$$

[4] This assumption is quite commonly used in this type of model. Bodie *et al.* (1992) studied the conditions under which this simplification is appropriate. Viceira (2001) examines portfolio choices using a model that includes uncertainty about labour income.

Using equation 2.4, we isolate P_{1t} as a function of R_{1t} and substitute in equation 2.3 in order to express the price of a one-period security as a function of the marginal rate of intertemporal substitution. As has been noted, the concept of absence of arbitrage allows us to replace $(R_{n+1})^i$ in equation 2.3 with securities of various maturities if their systematic risk is the same. Thus we obtain the following equation:

$$P_{1t} = E_t\left[B\left(\frac{C_{t+1}}{C_t}\right)^{-\alpha}\right] \tag{2.5}$$

which is equivalent to:

$$P_{1t} = E_t\{\exp[\log B - \alpha\Delta\log C_{t+1}]\} \tag{2.6}$$

where Δ is the first-difference operator and log is the natural log. Using the approximation that the rate of return r_{1t} = -log P_{1t}, we can apply a logarithmic transformation to equation 2.6, and obtain from a Taylor expansion:

$$r_{1t} = -\log P_{1t} = -\log B + \alpha E_t\Delta\log C_{t+1} - \frac{\alpha^2}{2}V_t(\Delta\log C_{t+1}) \tag{2.7}$$

where V_t represents the (time-varying) conditional variance of consumption growth ($\Delta\log C_{t+1}$).[5] This equation is similar to the one obtained by Campbell, Lo and MacKinlay (1997) under the joint assumptions of log-normality and homoskedasticity in the variance of consumption growth.[6] In equation 2.7, the (real) one-period interest rate is high if -log B is high (or if B is low). If the investor becomes more impatient, his time preference B declines (away from one and towards zero) and the interest rate increases in order to encourage him to defer his consumption to a later date. Finally, the one-period interest rate is high if the variance of consumption growth is low, since the investor has less need for precautionary savings: lower uncertainty about future consumption growth

[5] As shown in Lee (1995), the Taylor expansion also includes higher order terms that can be omitted if the change in expected growth of consumption is small.

[6] Log-normality is a simplifying assumption in models with constant variance, like in Campbell, Lo, and MacKinlay (1997).

means that the investor does not need to save as much as would be the case otherwise, and thus, a higher interest rate.

In order to calculate the rate of return on a security maturing in two periods (r_{2t}), let us first define the holding return as follows:

$$\begin{aligned} H_{2t+1} &= \frac{P_{2t+1} + D_{2t+1}}{P_{2t}} = \frac{1}{P_{2t}} \\ &= R_{2t}^2 \\ &= (1 + r_{2t})^2 \end{aligned} \tag{2.8}$$

where $P_{2t+1} = 1$ and $D_{2t+1} = 0$. R_{2t} represents the yield of a security maturing in two periods and held for one period, and R_{2t}^2 represents the yield of a security maturing in two periods and held for two periods. From equation 2.8, we isolate P_{2t} and substitute in the Euler equation, as follows:

$$P_{2t} = E_t\left[B^2 \left(\frac{C_{t+2}}{C_t} \right)^{-\alpha} \right] \tag{2.9}$$

which is equivalent to:

$$P_{2t} = E_t\{\exp[2\log B - \alpha\Delta\log C_{t+1} - \alpha\Delta\log C_{t+2}]\} \tag{2.10}$$

As is the case for the one-period security, the logarithmic form of equation 2.10 can be transformed by a Taylor expansion (with approximation that r_{2t} = -log P_{2t}) to obtain:

$$\begin{aligned} r_{2t} = \frac{-\log P_{2t}}{2} = -\log B + \frac{\alpha}{2} E_t \Delta\log C_{t+2} + \frac{\alpha}{2} E_t \Delta\log C_{t+1} - \frac{\alpha^2}{4} V_t\left(\Delta\log C_{t+2}\right) \\ - \frac{\alpha^2}{4} V_t\left(\Delta\log C_{t+1}\right) - \frac{\alpha^2}{2} Cov_t\left(\Delta\log C_{t+2}, \Delta\log C_{t+1}\right) \end{aligned} \tag{2.11}$$

We now have the equations for the rates of return of securities maturing in one and two periods, r_{1t} and r_{2t}.

2.3 The term premium

Assuming an absence of arbitrage between securities in the "spot market" and the "forward market", the rate of return of a one-period forward contract purchased at time t, beginning at time $t + 1$ and maturing at time $t + 2$, referred to as rf_{1t}, should be equal to the difference between the rate of return on a security maturing in two periods and held for two periods and the yield of a one-period security maturing in one period (see Appendix 1):

$$rf_{1t} = 2r_{2t} - r_{1t} \tag{2.12}$$

Furthermore, according to the Expectation Hypothesis of the Term Structure, the rate of return of a one-period forward contract is equal to the one-period ahead expected spot, referred to as $E_t r_{1t+1}$, plus a term premium φ_1 :

$$rf_{1t} = E_t r_{1t+1} + \varphi_1 \tag{2.13}$$

If we combine equations 2.12 and 2.13, we obtain the equation for the term structure of interest rates:

$$2r_{2t} = r_{1t} + E_t r_{1t+1} + \varphi_1$$

or

$$\varphi_1 = 2r_{2t} - r_{1t} - E_t r_{1t+1} \tag{2.14}$$

The first of these equations simply states that the rate of return on a security maturing in two periods ($2r_{2t}$) is the sum of the rates of return on securities maturing in one period purchased at time t (r_{1t}) and at time $t + 1$ ($E_t r_{1t+1}$), plus a term premium φ_1. If we insert equations 2.7 and 2.11 into equation 2.14, we obtain an expression for the term premium:

$$\varphi_{1t} = \frac{\alpha^2}{2} E_t V_{t+1}\left(\Delta \log C_{t+2}\right) - \frac{\alpha^2}{2} V_t\left(\Delta \log C_{t+2}\right) - \alpha^2 Cov_t\left(\Delta \log C_{t+2}, \Delta \log C_{t+1}\right) \tag{2.15}$$

Here, the term premium varies over time depending on the time-varying variance and covariance of consumption growth.

In order to make equation 2.15 more tractable, we need to identify the process that best describes consumption growth ($\Delta \log C_t$). Using quarterly Canadian data from 1960 to 2000, we find that the process most likely to reproduce the growth rate of per capita real consumption is an AR(1) for the mean equation and an ARCH(1) for the conditional variance equation:[7]

$$\Delta \log C_{t+1} = c + \rho \Delta \log C_t + \varepsilon_{t+1} \tag{2.16}$$

$$\varepsilon_{t+1}^2 = \upsilon_0 + \upsilon_1 \varepsilon_t^2$$

where $-1 < \rho < 1$, $\upsilon_0 > 0$ and $\upsilon_1 > 0$. The degree of persistence in consumption growth is indicated here by ρ.

We can then replace the process described in equation 2.16 in each of the three terms of equation 2.15. The first term is:

$$\begin{aligned} E_t V_{t+1}(\Delta \log C_{t+2}) &= E_t E\left[(\Delta \log C_{t+2} - E_{t+1} \Delta \log C_{t+2})^2\right] \\ &= E_t E\left[\varepsilon_{t+2}^2\right] \\ &= \sigma_{c,t+2}^2 \end{aligned}$$

where $\sigma_{c,t+2}^2$ represents the conditional variance of consumption growth at time $t + 2$. The second term is:

$$\begin{aligned} V_t(\Delta \log C_{t+2}) &= E\left[(\Delta \log C_{t+2} - E_t \Delta \log C_{t+2})^2\right] \\ &= E\left[(\varepsilon_{t+2} + \rho \varepsilon_{t+1})^2\right] \\ &= \rho^2 \sigma_{c,t+1}^2 + \sigma_{c,t+2}^2. \end{aligned}$$

The third term is:

[7] Greater detail about the estimation results of this model is provided below.

$$Cov_t(\Delta \log C_{t+2}, \Delta \log C_{t+1}) = E[(\varepsilon_{t+2} + \rho\varepsilon_{t+1})(\varepsilon_{t+1})]$$
$$= \rho\sigma^2_{c,t+1}.$$

Replacing these three terms in equation 2.15 gives us:

$$\varphi_{1t} = \frac{-\alpha^2}{2}(\rho^2 + 2\rho)\sigma^2_{c,t+1}. \qquad (2.17)$$

Equation 2.17 provides us with an expression for the term premium, which depends on three components: the degree of risk aversion α, the persistence of consumption growth ρ and the conditional variance of consumption growth $\sigma^2_{c,t+1}$. The sign of the term premium depends on the signs of its components. Knowing that α and $\sigma^2_{c,t+1}$ are positive and that $-1 < \rho < 1$, the term premium is positive if the growth rate of consumption is negatively autocorrelated ($\rho < 0$), and negative if the consumption growth rate is positively autocorrelated. For example, a negative value for ρ reduces the probability of having consumption at time $t + 2$ deviate even further from its expected path following a shock at time $t + 1$. This lowers precautionary savings held in the form of two-period securities, reducing demand for these securities while increasing their rate of return as well as the term premium.[8] Equation 2.17 also shows that the term premium varies over time since the consumption growth rate is conditionally heteroskedastic. The size of the term premium varies directly with the variance of consumption growth (when $\rho < 0$): for example, greater uncertainty about the growth rate of future consumption increases demand for one-period securities as opposed to demand for two-period securities (because of the preference for liquidity) and thus increases the spread between two-period and one-period interest rates.

2.4 The theoretical model and the data

Our goal here is to measure the size of the term premium and its variability obtained in the theoretical model using Canadian consumption data. Table 2 shows the

estimation results obtained using the AR-ARCH model of the growth rate of per capita real consumption.[9]

Table 2

Estimation results of $\Delta \log C_{t+1} = c + \rho \Delta \log C_t + \varepsilon_{t+1}$, and $\varepsilon_{t+1}^2 = \upsilon_0 + \upsilon_1 \varepsilon_t^2$

Sample : 1961Q3 to 2000Q3

	c	ρ	υ_0	υ_1
Coefficient	0.00471	-0.20475	0.00002	0.55739
(z-stat)	(13.75)	(-2.04)	(4.70)	(3.53)

When we replace the estimated coefficient ρ and the sample variance of consumption growth σ^2 in the term premium equation 2.17, we obtain an average theoretical term premium varying from 1 to 123 basis points for risk aversion coefficients ranging between 1 and 10, as shown in Table 3. For a risk aversion coefficient of 5, the average theoretical term premium is 31 basis points, which corresponds to the mean of the excess returns calculated using three-month Treasury bill data from 1960 to 2000.

[8] This relationship is provided by equation 2.11, which shows that the rate of return on securities maturing in two periods depends on the covariance between the consumption growth rate at time $t+2$ and at time $t+1$.

[9] Per capita consumption is defined as the sum of consumption of non-durable goods (D14845), semi-durable goods (D14844), and services (D14846), divided by Canada's total population (D1).

Table 3
Theoretical term premium, by risk aversion coefficient (%)

Coefficient of risk aversion	Average term premium (φ_1)
1	0.000124
2	0.000494
3	0.001112
4	0.001977
5	0.003089
7	0.006054
10	0.012355

Chart 2 (Appendix 3) shows, for a risk aversion coefficient of 5, the changes in the theoretical term premium over the 1960-2000 period, using the conditional variance obtained from the estimated ARCH process of consumption growth. The term premium is usually rather small, fluctuating between 20 and 50 basis points, except for certain periods when it showed very marked but short-lived increases, such as those during the 1981-82 and 1990-91 recessions. Chart 2 also shows the estimated changes in the theoretical term premium for a risk aversion coefficient of 1.5, which reproduces the average real interest rate over the 1960-2000 (see Appendix 2). For this risk aversion coefficient, the level and variability of the term premium are very small.

We could also use alternative measures of consumption to calculate the theoretical term premium.[10] However, Campbell and Cochrane (2000) emphasize that the empirical difficulties of the consumption-based asset pricing models extend beyond errors in measuring consumption. A further possible way of estimating changes in the term premium would be to estimate the AR-ARCH consumption equation, and its coefficient ρ, recursively. We did re-estimate equation 2.16 in each quarter for a moving

[10] For example, we could increase the variance of consumption growth (and that of the term premium) by adding the proportion of spending on durable goods that is consumer spending; or we could exclude the proportion of spending on services, such as housing, that cannot be compressed for most households.

sample of 40 quarters, and found considerable change in the coefficient ρ over the period.[11]

Overall, if selected risk aversion coefficients are used, the theoretical model can reproduce the average term premium observed in Canadian data since the 1960s, and a large but temporary variability in the estimated premium. However, in order to know if the model accurately reproduces the changes in the term premium, it must be tested empirically, which we do in the next section.

3. Empirical testing of the theoretical model

In this section, we use an econometric equation to test the theoretical model, by verifying the effect on excess returns of the conditional variance of consumption growth and other macro-economic variables. The econometric equation takes into account changes in the term premium, as estimated by the economic variables selected, and tests whether these changes are adequate to ensure that the Expectation Hypothesis of the Term Structure (EHTS) is not rejected. In many ways, our econometric equation is similar to that used in Hejazi, Lai and Yang (2000), in particular the role played by macro-economic variables.

3.1 The econometric equation

It is possible to include in the EHTS equation 1.5 (presented in Section 1) economic variables explaining changes in the term premium and to test the EHTS, $\gamma_1 = 0$, while taking the effect of these variables into account. In order to examine this, we posit an equation in the following form:

$$[rf_{1t} - r_{1t+1}] = \gamma_0 + \gamma_1[rf_{1t} - r_{1t}] + \gamma_x X + \mu_{t+1} \qquad (3.1)$$

[11] Although this coefficient has been quite stable around -0.20 since the early 1990s, it was positive from the beginning of the 1981-82 recession until the mid-1980s, which suggests that the term premium was negative during that period because of the large demand for long-term securities.

where X represents a vector of variables that may help explain changes in the term premium.[12] The studied variables are the following:

$\sigma^2_{c,t+1}$ represents the conditional variance of the growth rate of per capita real consumption;

$\sigma^2_{r,t}$ represents the conditional variance of excess returns;

$\sigma^2_{\pi,t-1}$ represents the conditional variance of the Consumer Price Index inflation rate (excluding food, energy, and the effects of indirect taxation);

$\sigma^2_{s,t-1}$ represents the conditional variance of changes in the nominal exchange rate for the Canadian dollar; and

gdy_{t-1} represents the federal government debt-to- nominal GDP ratio.[13]

The conditional variances σ^2_c, σ^2_r, σ^2_π and σ^2_s measure respectively, economic uncertainty on: the growth rate of consumption, the interest rate, the inflation rate, and changes in the exchange rate. The conditional variances σ^2_c, σ^2_π and σ^2_s are estimated using univariate ARMA-GARCH models, and then calculated for each variable. The conditional variance of excess returns σ^2_r is obtained directly from the estimation of equation 3.1, which, as we pointed out before, includes a GARCH process.[14]

In Section 2, we discussed in detail the role of uncertainty about future consumption growth as a determinant of the term premium. The conditional variance of

[12] Given that the vector X contains mainly generated regressors in the following analysis, one has to take with caution the statistical inference done in this Section.

[13] We use in equation 3.1 a measure of federal government debt (Cansim, B2501) divided by Gross Domestic Product in current dollars (Cansim, D14816). In preliminary work, we used an annual series of net public debt (including debt of the federal government, provinces and municipalities), but we had to construct the quarterly data for the estimation of equation 3.1. The estimation results were similar to those presented in this paper.

[14] Another possibility would be to estimate equation 3.1 and the conditional variances simultaneously using a multivariate ARMA-GARCH model, as in Lee (1995). However, our calculations show that the correlation between the error terms of the univariate ARMA-GARCH models is very small (not exceeding 0.14), suggesting that the efficiency gain from a multivariate model could also be relatively small.

excess returns σ_r^2 measures uncertainty about future short-term interest rates. Since the study by Engle, Lilien and Robins (1987) was published, several papers have shown that the variance of interest rates, estimated using a GARCH process, could help explain the term premium. Uncertainty about future inflation may affect the term premium if expectation errors about a nominal variable, such as overall prices or inflation, can affect real consumption decisions. The variability in the future exchange rate is an additional source of uncertainty in an open economy with a floating exchange rate like that of Canada since 1972.

In equation 3.1, the term premium also depends on the federal government debt-to-nominal GDP ratio *gdy*. We assume that the government debt-to-GDP ratio represents an estimate for the risk premium embodied in longer-term interest rates to compensate investors for the inflationary risk of monetization or the risk of future taxation often associated with public indebtedness.[15] We should note that the variable *gdy* is non-stationary according to the usual Dickey-Fuller (D-F) and Phillips-Perron (P-P) unit-root tests. Equation 3.1 includes the variable in level form and in first difference. Although the variable *gdy*, which is a I(1) variable, helps explain changes in excess returns, a I(0) variable according to the usual D-F and P-P unit-root tests, this relation is not statistically spurious in our opinion. Instead, we think it can be explained by the fact that the series of excess returns has a small non-stationary component (possibly related to the government debt-to-GDP ratio) that is difficult to identify using the usual unit-root tests, given that the variance of the series is dominated by a stationary component (mainly related to expectation errors).

3.2 The estimation results

Table 4 presents the estimation results obtained using equation 3.1 for the period beginning in the first quarter of 1973 and ending in the third quarter of 2000, a period in which the Canadian dollar follows a floating exchange rate regime. According to the

[15] See Macklem (1994-95) and Clinton (1998) for a discussion about the effects of public debt on interest rates.

results of equation 3.1a, the EHTS is rejected for this period since the term spread $rf_{1t} - r_{1t}$ significantly affects excess returns. This also corresponds to the results presented in Table 1.

Equation 3.1b indicates that the conditional variance of consumption growth σ_c^2 is statistically significant in explaining changes in excess returns, thus showing that uncertainty about future consumption growth is an important factor in explaining the term premium, as suggested in our theoretical model. That said, although the variable σ_c^2 is statistically significant, the simple fact that it is used in the model does not prevent rejection of the EHTS hypothesis that $\gamma_1 = 0$.

Equation 3.1c indicates that none of the conditional variances σ_r^2, σ_π^2 and σ_s^2 are significant in the equation for excess returns, thus showing that uncertainties about the interest rate, the inflation rate, and the changes in the exchange rate are not important factors in explaining the term premium during our sample period. These results are not very different from those of Hejazi, Lai and Yang (2000), except for the fact that this study found the conditional variance of interest rates significant in some equations.

To obtain equation 3.1d, we add to equation 3.1b the federal debt-to-GDP ratio *gdy* and its first difference. We find that the coefficient on the level of *gdy* is clearly significant, but that the coefficient on the first difference of *gdy*, although higher, is less significant statistically. In addition, we note that the term spread is no longer significant in equation 3.1d and thus that the EHTS is not rejected in this instance. As a result, the term spread is not included in equation 3.1e.

Table 4
Tests of term premium determinants
Estimated equation:

$$[rf_{1t} - r_{1t+1}] = \gamma_0 + \gamma_1[rf_{1t} - r_{1t}] + \gamma_x X + \mu_{t+1}$$

Sample period: 1973Q1 to 2000Q3

	Estimated coefficients and z-statistics				
Dependant variables (X)	Eq. 3.1a	Eq. 3.1b	Eq. 3.1c	Eq. 3.1d	Eq. 3.1e
γ_0	0.0013	0.0002	0.0020	-0.0053	-0.0050
	(1.36)	(0.17)	(0.83)	(-2.81)	(-2.68)
$rf_{1t} - r_{1t}$	0.56	0.53	0.46	0.20	
	(3.71)	(3.12)	(3.27)	(1.03)	
$\sigma^2_{c,t-1}$		3.22	3.43	1.97	1.51
		(5.30)	(6.73)	(2.70)	(2.27)
$\sigma^2_{r,t}$			-4.38		
			(-0.72)		
$\sigma^2_{\pi,t-1}$			-2.65		
			(-0.50)		
$\sigma^2_{s,t-1}$			-0.17		
			(-0.81)		
Δgdy_t				0.13	0.13
				(1.63)	(1.75)
gdy_{t-1}				0.017	0.019
				(3.08)	(3.75)
Proportion of the variance of excess returns explained by variables X other than the term spread	-	0.036	0.070	0.065	0.061

We now use equation 3.1e to calculate the term premium during our sample period, the result being presented in Chart 3 (Appendix 3). The term premium varied between slightly negative values in the mid-1970s and maximum values of about 70 basis points during the first half of the 1990s (excluding a peak of 150 basis points in the third quarter of 1991). The variability of the term premium can largely be explained by changes in the government debt-to-GDP ratio, while the effects of the conditional variance of consumption growth were mainly episodic, being felt particularly during the

1981-82 and 1990-91 recessions. As well, the results indicate that the term premium has dropped steadily by about 50 basis points since 1995 following the decline in the government debt-to-GDP ratio. The term premium stands at about 20 basis points at the end of 2000.

Furthermore, it is interesting to use the forward interest rate, adjusted for the term premium, to evaluate future interest rates recently anticipated by financial markets. Chart 4 presents the actual three-month Treasury bill interest rate and the corresponding interest rate anticipated by the markets three months in advance. The markets did a relatively good job of predicting the increase in Canadian short-term interest rates that followed the emerging Asian crisis in the fall of 1997 and the financial problems in Russia and Brazil in the summer of 1998. As well, in a more stable macro-economic environment, the markets did a good job of predicting the tightening in monetary policy and higher interest rates that occurred from the fall of 1999 to the summer of 2000. That said, in late 2000 and early 2001, the markets were anticipating lower interest rates in the months to come, but interest rates dropped more quickly and more markedly than anticipated.

4. Conclusions

Quite often the Expectation Hypothesis of the Term Structure (EHTS) is used to explain changes in forward interest rates. The EHTS assumes that forward rates represent future interest rates, as anticipated by economic agents, plus a term premium. That said, it is well known that the EHTS is often rejected by the data, perhaps because the term premium varies over time.

In this paper, we presented and tested a theoretical model aimed at explaining the term premium included in the three-month forward interest rate. In this model, the premium depends on two parameters — the degree of risk aversion and the degree of persistence in consumption growth — and on one variable, the conditional variance of (or the uncertainty associated with) consumption growth. When we tested the model, we

found that the conditional variance of consumption growth is statistically significant in an econometric equation of excess returns, but the simple fact that it appears in the equation does not prevent the rejection of the EHTS. However, the EHTS is not rejected in an equation that includes the conditional variance of consumption growth and a measure of the federal government debt-to-GDP ratio as a proxy for the time-varying risk premium often associated with changes in government debt. Using this broader equation, we found that the term premium for the three-month forward interest rate was large and variable during most of the 1980s and 1990s, but has dropped steadily since 1995 following the decline in the government debt-to-GDP ratio. Combined with the reduction in long-term interest rates brought about by lower government debt, this reduction in short-term rates represents considerable savings in debt service over time

This important feature of the term premium coupled with the finding that EHTS cannot be rejected, enables us to use current market interest rates to evaluate expectations of three-month interest rates, information that can be useful in conducting economic policy.

Two avenues for future research appear promising. First, some recent papers (Hejazi, Lai, Yang 2000, and Gravelle and Moessner 2001) have shown that the short-term variability of Canadian interest rates is influenced mainly by shocks or changes affecting macro-economic variables in the United States. It might be interesting to broaden our study to include the conditional variance of U.S. variables. Second, where estimating techniques are concerned, we might want to use the Kalman filter methodology, which is appropriate for estimating models using unobservable variables, such as the term premium (see Gravelle and Morley 2000).

References

Backus, David; Foresi, Silverio; Mozumdar, Abon; Wu, Liuren (2001) "Predictable Changes in Yields and Forward Rates." *Journal of Financial Economics*, 59, pp. 281-311.

Backus, David; Gregory, Allan; Zin, Stanley (1989) "Risk Premiums in the Term Structure: Evidence from Artificial Economies." *Journal of Monetary Economics*, 24, pp. 371-399.

Bodie, Zvi; Merton, Robert C.; Samuelson, William F. (1992) "Labor Supply Flexibility and Portfolio Choice in a Life Cycle Model." *Journal of Economics Dynamics and Control*, Vol. 16, No. 3-4, pp. 427-49.

Campbell, John (1986) "Bond and Stock Returns in a Simple Exchange Model." *Quarterly Journal of Economics*, Vol. 101, pp. 786-803.

Campbell, John; Cochrane, John (2000) "Explaining the Poor Performance of Consumption-based Asset Pricing Models." *The Journal of Finance*, Vol.55, No.6, pp. 2863-2878.

Campbell, John; Lo, Andrew; MacKinlay, A. Craig (1997) "*The Econometrics of Financial Markets*." Princeton University Press, Princeton, New Jersey, 611 pages.

Campbell, John; Viceira, Luis, M. (2001) "Who Should Buy Long-Term Bonds?", *The American Economic Review*, Vol. 91, No. 1, pp. 99-127.

Clinton, Kevin (1998) "Canada-U.S. long-term interest differentials in the 1990s" *Bank of Canada Review*, Spring, pp. 17-38.

Engle, R.F.; Lilien, D.M.; Robins, R.P. (1987) "Estimating Time-Varying Risk Premia in the Term Structure: the ARCH-M Model." *Econometrica*, Vol. 55, No. 2, pp391-407.

Gravelle, Toni; Morley, James C. (2000) "New Findings on the Expectations Hypothesis and the Canadian Term Structure of Interest Rates". Paper presented at the annual meeting of the Canadian Economics Association, Vancouver, June 2000.

Gravelle, Toni; Moessner, Richhild (2001) "Reactions of Canadian Interest Rates to Macroeconomic Announcements: Implications for Monetary Policy Transparency." Bank of Canada, Working Paper 2001-5.

Hejazi, Walid; Lai, Huiwen; Yang, Xian (2000) "The expectations hypothesis, term premia, and the Canadian term structure of interest rates." *Canadian Journal of Economics*, Vol.33, No.1, pp. 133-148.

Lee, Sang-Sub (1995) "Macroeconomic Sources of Time-Varying Risk Premia in the Term Structure of Interest Rates." *Journal of Money, Credit, and Banking*, Vol 27, No 2, pp. 549-569.

Ouellette, Pierre; Paquet, Alain (1999) "Inflation uncertainty and the cost of inflation." *Canadian Journal of Economics*, Vol.32, No.1, pp. 195-214.

Macklem, Tiff (1994-95) "Some macroeconomic implications of rising levels of government debt." *Bank of Canada Review*, Winter, pp. 41-60.

Paquette, Jean-Yves; Stréliski, David (1998) "L'utilisation des accords de taux futur (FRA) [Forward Rate Agreements] au Canada". *Bank of Canada Review*, Spring.

Viceira, Luis M (2001) "Optimal Portfolio Choice for Long-Horizon Investors with Nontradable Labor Income." *Journal of Finance*, Vol 56, No. 2, pp. 433-70.

Appendix 1

Absence of arbitrage and the calculation of the implicit forward interest rate

If financial markets are efficient, there should be no arbitrage at the equilibrium between the rate of return of a long-term security on the "spot market" and the rate of return obtained by investing on short-term securities on the spot and the forward markets over the same time period. In other words, the long-term interest rate should be equal to the geometric average of the spot and the forward short-term interest rates. The long-term interest rate can be represented as follows:

$$(1+r_n) = [(1+r_1)(1+rf_1)(1+rf_2)(1+rf_3)...(1+rf_{n-1})]^{1/n} \qquad (A.1)$$

where r_n is the interest rate for a long-term security of n periods (or quarters) on the spot market; r_1 is the interest rate for a one-period security on the spot market; and rf_1 to rf_{n-1} are the forward interest rates for one-period securities for all the next n periods.

Equation A.1 can be rewritten to isolate the forward interest rate. For example, the implicit one-period forward interest rate for a security maturing in two periods can be obtained using the interest rates on the spot market for one-period and two-period investments, as follows:

$$rf_1 = [(1+r_2)^2/(1+r_1)] - 1 \qquad (A.2)$$

When interest rates are relatively low, the forward interest rate can be approximated as follows:

$$rf_1 = 2r_2 - r_1 \qquad (A.3)$$

Equation A.3 is used in Section 2 (equation 2.12) to obtain the theoretical model for the term premium.

Appendix 2

Calculation of the theoretical real interest rate

One way of verifying the acceptability of the theoretical model presented in Section 2 is to check whether it accurately reproduces the average three-month real interest rate found in Canadian data. Table A1 presents the theoretical values of r_1 calculated using equation 2.7, assuming reasonable time preference and risk aversion coefficients B and α.

Table A1
Theoretical three-month real interest rate (%)

	Rate of time preference (B)	
	0.99	0.995
Coefficient of risk aversion		
1	0.027	0.022
1.5	0.035	0.030
2	0.043	0.038
3	0.058	0.053
4	0.073	0.068
5	0.087	0.082

Given: (i) the average three-month real interest rate of 3.3% during the period from 1960 to 2000; (ii) the growth rate of per capita real consumption of 1.7%; and (iii) the variance of consumption growth of 0.067%, the model requires a risk aversion coefficient of approximately 1.5 in order to reproduce the average three-month real interest rate r_1 when time preference B is somewhere between 0.99 and 0.995.[16]

[16] Ouellette and Paquet (1999) indicate an estimated time preference of 0.9986 with data expressed at a quarterly rate, which is equivalent to a time preference of 0.9944 with data expressed at an annual rate, such as the data used in this paper.

Appendix 3

Chart 1
Excess returns and term spread on three-month securities (%)

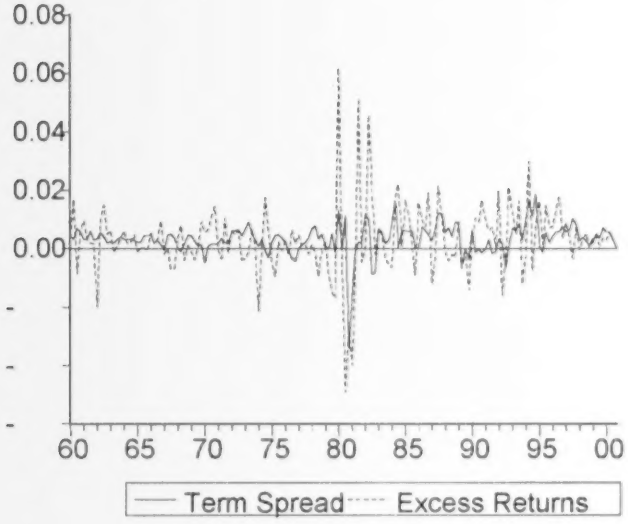


Chart 2
Theoretical term premium, by risk aversion coefficient (%)

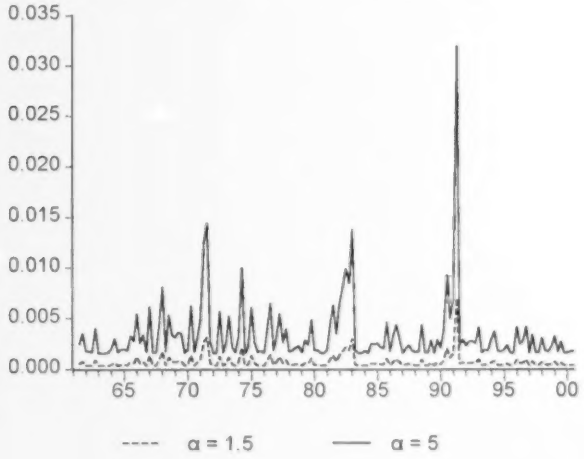
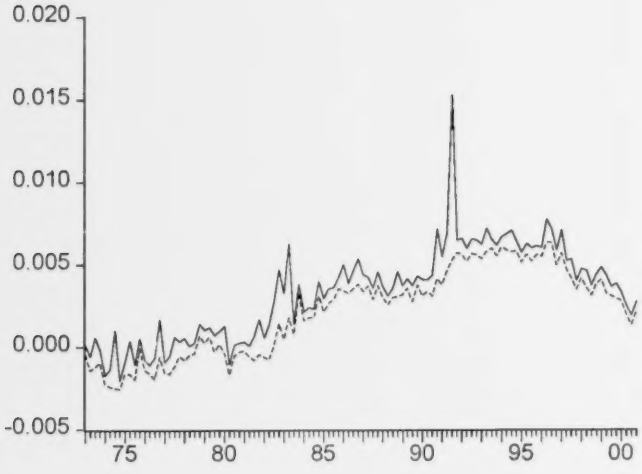


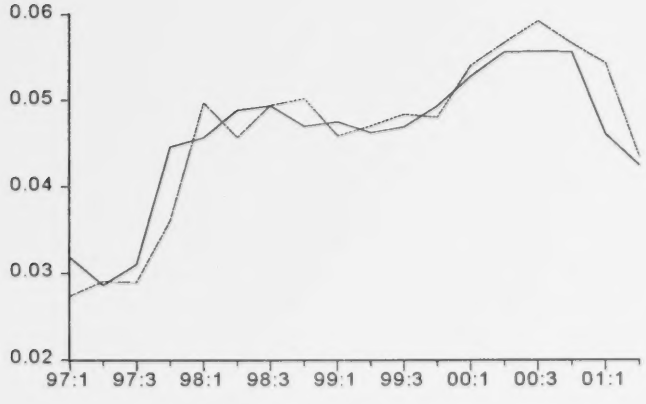
Chart 3
Term premium, estimated using econometric model (%)



---- Component of the term premium that can be attributed to changes in the federal government debt-to-GDP ratio

—— Total term premium including the effects of government debt-to-GDP ratio and variance of consumption growth

Chart 4
Expected and realized three-month T-Bill rates (%)



—— Realized 3-month interest rate.

------ Expected 3-month interest rate, three months ahead.